Tutorial 5

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6th week

1. Find the average value of ρ over the solid sphere $\rho \leqslant a$.

Solid sphere

$x = \rho \sin\phi \cos\theta$ $\quad \rho \in [0, a]$

$y = \rho \sin\phi \sin\theta$ $\quad \theta \in [0, 2\pi)$

$z = \rho \cos\phi$ $\quad \phi \in [0, \pi]$

$$\int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \sin\phi \left. \frac{\rho^4}{4} \right|_0^a d\phi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi} \sin\phi \frac{a^4}{4} d\phi \, d\theta$$

$$= \frac{a^4}{4} \int_0^{2\pi} \left. -\cos\phi \right|_0^{\pi} d\theta$$

$$= \frac{a^4}{4} \int_0^{2\pi} 2 \, d\theta$$

$$= a^4 \pi$$

Average Value $= \frac{1}{V} \int_0^{2\pi} \int_0^{\pi} \int_0^{a} \rho^2 \cdot \rho \sin\phi \, d\rho \, d\phi \, d\theta.$

$$= \frac{1}{\frac{4}{3} a^3 \pi} a^4 \pi = \frac{3}{4} a$$

2. Convert to cylindrical coordinates. Then evaluate the new integral.

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-(x^2+y^2)}^{(x^2+y^2)} 21xy^2 \mathrm{d}z \mathrm{d}y \mathrm{d}x$$

Let $x = r\cos\theta \quad y = r\sin\theta \quad z = z. \quad r \geq 0 \quad \theta \in [-\pi, \pi)$

$$\left.\begin{array}{l} 0 \leq x \leq 1 \\ -\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2} \end{array}\right\} \Rightarrow \left.\begin{array}{l} 0 \leq x \leq 1 \\ x^2+y^2 \leq 1 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 0 \leq r \leq 1 \\ 0 \leq r\cos\theta \leq 1 \end{array}\right\} \begin{array}{l} 0 \leq r \leq 1 \\ -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \end{array}$$

$$-(x^2+y^2) \leq z \leq x^2+y^2 \Rightarrow -r^2 \leq z \leq r^2$$

$$\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-(x^2+y^2)}^{x^2+y^2} 21xy^2 \, dz \, dy \, dx$$

$$= 21 \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-r^2}^{r^2} r\cos\theta \, r^2 \sin^2\theta \, r \, dz \, d\theta \, dr$$

$$= 42 \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^6 \cos\theta (1 - \cos^2\theta) \quad d\theta \, dr$$

$$= 42 \int_0^1 r^6 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta - \frac{1}{4}(\cos 3\theta + 3\cos\theta) \, d\theta \, dr$$

$$= 42 \int_0^1 r^6 \left[-\frac{1}{12}\sin 3\theta + \frac{1}{4}\sin\theta \right] \Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}} dr$$

$$= 42 \int_0^1 r^6 \left[\left(\frac{1}{12} + \frac{1}{4}\right) - \left(-\frac{1}{12} - \frac{1}{4}\right) \right] dr$$

$$= 42 \int_0^1 r^6 \cdot \frac{2}{3} \, dr$$

$$= 28 \cdot \frac{1}{7} r^7 \Big|_0^1 = 4$$

3. Set up an integral in rectangular coordinates equivalent to the integral

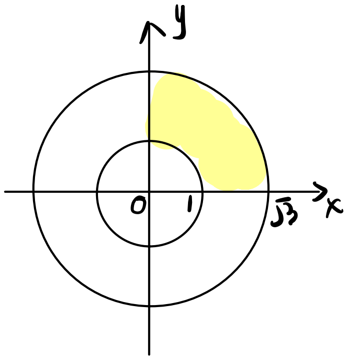
$$\int_0^{\pi/2} \int_1^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r^3(\sin\theta\cos\theta)z^2 \mathrm{d}z\mathrm{d}r\mathrm{d}\theta.$$

Arrange the order of integration to be z first, then y, then x.

$x = r\cos\theta \quad y = r\sin\theta \quad z = z$

$$\left.\begin{array}{l} 1 \le r \le \sqrt{3} \\ 0 \le \theta \le \frac{\pi}{2} \end{array}\right\} \Rightarrow \left.\begin{array}{l} 0 \le x \le \sqrt{3},\ y \ge 0 \\ 1 \le x^2+y^2 \le 3 \end{array}\right\} \Rightarrow \begin{array}{l} 1 \le x \le \sqrt{3} \\ \sqrt{1-x^2} \le y \le \sqrt{3-x^2},\ y \ge 0 \end{array}$$

$$1 \le z \le \sqrt{4-r^2} \Rightarrow 1 \le z \le \sqrt{4-x^2-y^2}$$

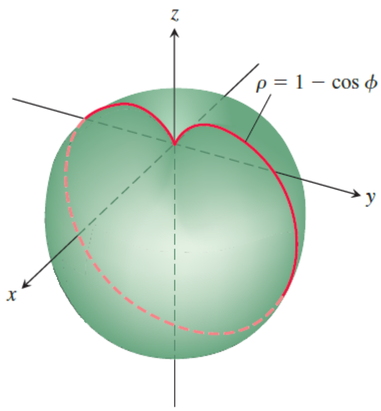


$$\int_0^{\frac{\pi}{2}} \int_1^{\sqrt{3}} \int_1^{\sqrt{4-r^2}} r^3(\sin\theta\cos\theta)z^2 \,dz\,dr\,d\theta$$

$$= \int_0^1 \int_{\sqrt{1-x^2}}^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} xyz^2 \,dz\,dy\,dx$$

$$+ \int_1^3 \int_0^{\sqrt{3-x^2}} \int_1^{\sqrt{4-x^2-y^2}} xyz^2 \,dz\,dy\,dx$$

4. Find the moment of inertia about the z-axis of a solid of density $\delta = 1$ enclosed by the spherical coordinate surface $\rho = 1 - \cos\phi$. The solid is the red curve rotated about the z-axis in the accompanying figure.



The region is enclosed by

$0 \le \rho \le 1 - \cos\phi$ $\quad x = \rho \sin\phi \cos\theta$

$0 \le \phi \le \pi$ $\quad y = \rho \sin\phi \sin\theta$

$0 \le \theta \le 2\pi$ $\quad z = \rho \cos\phi$

$$I_z = \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi} (x^2+y^2)\,\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi} \rho^4 \sin^3\phi \, d\rho\, d\phi\, d\theta$$

$$= \frac{1}{5}\int_0^{2\pi}\int_0^{\pi} (1-\cos\phi)^5 \sin^3\phi \, d\phi\, d\theta$$

$$= \frac{64}{35}\pi.$$

5. What relationship must hold between the constants a, b, and c to make

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(ax^2+2bxy+cy^2\right)}dxdy = 1?$$

(Hint: Let $s = \alpha x + \beta y$ and $t = \gamma x + \delta y$, where $(\alpha\delta - \beta\gamma)^2 = ac - b^2$. Then $ax^2 + 2bxy + cy^2 = s^2 + t^2$.)

$$x = \frac{\beta t - \delta s}{\beta\gamma - \alpha\delta}$$

$$y = \frac{\gamma s - \alpha t}{\beta\gamma - \alpha\delta}$$

$$\frac{\partial(x,y)}{\partial(s,t)} = \frac{1}{(\beta\gamma - \alpha\delta)^2}\begin{vmatrix} -\delta & \beta \\ \gamma & -\alpha \end{vmatrix} = \frac{1}{\alpha\delta - \beta\gamma}$$

$$\underline{I} = \int_{-\infty}^{+\infty}\int_{-\infty}^{\infty} e^{-(ax^2+2bxy+cy^2)}dxdy$$

$$= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(s^2+t^2)}\frac{\partial(x,y)}{\partial(s,t)}dsdt$$

$$= \frac{1}{\alpha\delta - \beta\gamma}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-(s^2+t^2)}dsdt$$

$$= \frac{1}{\alpha\delta - \beta\gamma}\cdot\sqrt{\pi}\cdot\sqrt{\pi} = 1$$

$$\alpha\delta - \beta\gamma = \pi$$

So $ac - b^2 = \pi^2$